Please check the examination details below before entering your candidate information

Candidate surname	Other names

Pearson Edexcel
Level 3 GCE

Centre Number ☐☐☐☐☐ Candidate Number ☐☐☐☐

Specimen Paper

(Time: 1 hour 40 minutes) Paper Reference **8FM0/01**

Further Mathematics

Advanced Subsidiary
Paper 1: Core Pure Mathematics

You must have:
Mathematical Formulae and Statistical Tables, calculator

Total Marks

Candidates may use any calculator allowed by the regulations of the Joint Council for Qualifications. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 9 questions in this question paper. The total mark for this paper is 80.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over ▶

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Pearson

Answer ALL questions. Write your answers in the spaces provided.

1.

$$\mathbf{P} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \qquad \mathbf{Q} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

The matrices **P** and **Q** represent linear transformations, P and Q respectively, of the plane.

The linear transformation M is formed by first applying P and then applying Q.

(a) Find the matrix **M** that represents the linear transformation M.

(2)

(b) Show that the invariant points of the linear transformation M form a line in the plane, stating the equation of this line.

(3)

1. a) Matrix M represents

transformation P, followed by transformation Q

Transformation A followed by B is represented by the matrix BA

$\therefore\ M = QP$

MATRIX MULTIPLICATION:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

$$M = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix} \times \frac{1}{2}$$

$$= \begin{bmatrix} -1(1)+0(-\sqrt{3}) & -1(\sqrt{3})+0(1) \\ 0(1)+1(-\sqrt{3}) & 0(\sqrt{3})+1(1) \end{bmatrix} \times \frac{1}{2}$$

$$= \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}$$

Question 1 continued

any invariant point as it is unchanged by the transformation

b) Invariant points → $M\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\\y\end{bmatrix}$

$$\begin{bmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix}=\begin{bmatrix}u\\v\end{bmatrix}$$

invariant means the same

$$\begin{bmatrix}-\frac{u}{2}-\frac{\sqrt{3}}{2}v\\ -\frac{\sqrt{3}}{2}u+\frac{v}{2}\end{bmatrix}=\begin{bmatrix}u\\v\end{bmatrix}$$

Set like components equal:

$$-\frac{u}{2}-\frac{\sqrt{3}}{2}v=u \qquad -\frac{\sqrt{3}u}{2}+\frac{v}{2}=v$$

$$-\frac{\sqrt{3}}{2}v=\frac{3u}{2} \qquad -\frac{\sqrt{3}u}{2}=\frac{v}{2}$$

$$-\sqrt{3}\overset{\overset{x}{\downarrow}}{u}=\overset{\overset{y}{\downarrow}}{v}$$

$$y=-\sqrt{3}x$$

(Total for Question 1 is 5 marks)

2. (a) Sketch, on an Argand diagram, the set of points

$$X = \{z \in \mathbb{C} : |z - 4 - 2\mathrm{i}| < 3\} \cap \left\{z \in \mathbb{C} : 0 \leqslant \arg(z) \leqslant \frac{\pi}{4}\right\}$$

On your diagram

- shade the part of the diagram that is included in the set
- use solid lines to show the parts of the boundary that are included in the set, and use dashed lines to show the parts of the boundary that are not included in the set

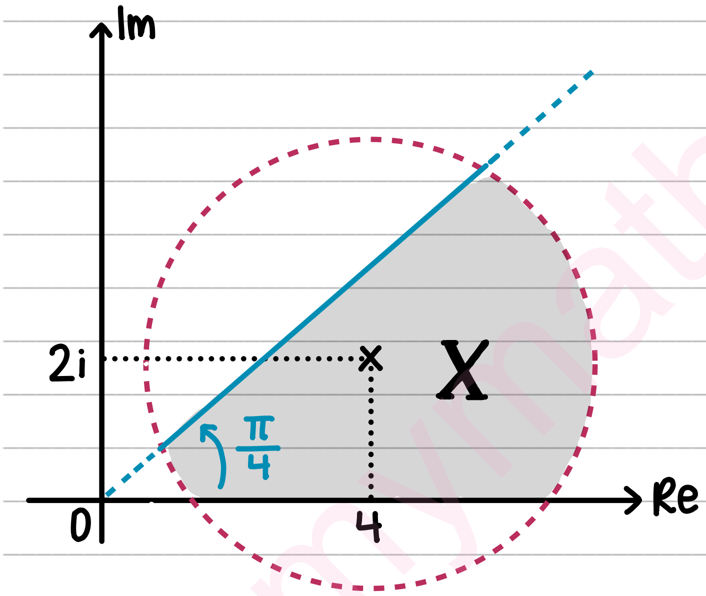
(3)

(b) Show that the complex number $z = 5 + 4\mathrm{i}$ is in the set X.

(3)

2.a) $X = \{z \in \mathbb{C} : |z - (4+2i)| < 3\}$ → circle with centre $(4,2)$ and radius < 3

AND → $\cap \{z \in \mathbb{C} : 0 \leq \arg(z) \leq \frac{\pi}{4}\}$



b) $|z - (4+2i)| < 3$

$\hookrightarrow |(5+4i) - 4 + 2i| < 3$

$|1 + 2i| < 3$

$1^2 + 2^2 < 9 \longrightarrow 5 < 9 \therefore 5 + 4i$ is in circle

Question 2 continued

$0 \leq \arg(z) \leq \frac{\pi}{4}$

$\hookrightarrow \arg(5+4i) = \tan^{-1}\left(\frac{4}{5}\right) = 0.6747\ldots < \frac{\pi}{4}$

$\therefore 0 \leq \arg(5+4i) \leq \frac{\pi}{4}$

As $5+4i$ is part of both loci, $5+4i \in X$

(Total for Question 2 is 6 marks)

Turn over ▶

3. (a) Find, in terms of the real constant k, the determinant of the matrix

$$\mathbf{M} = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & -1 \\ 1 & k & 2 \end{pmatrix}$$

(2)

Three distinct planes, Π_1, Π_2 and Π_3, are defined by the equations

$$\Pi_1 \ : \ \mathbf{r} \cdot \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = 4$$

$$\Pi_2 \ : \ \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$\Pi_3 \ : \ x + ky + 2z = -1$$

where λ and μ are scalar parameters.

(b) Find an equation in Cartesian form for

(i) Π_1

(ii) Π_2

(4)

Given that the three planes Π_1, Π_2 and Π_3 form a sheaf,

(c) use the answer to part (a) to explain why $k = -1$

(2)

3a) $M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \longrightarrow \det(M) = |M| = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

$$M = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & -1 \\ 1 & K & 2 \end{bmatrix}$$

$\longrightarrow \det(M) = |M| = 3\begin{vmatrix} 3 & -1 \\ K & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 3 \\ 1 & K \end{vmatrix}$

$= 3\left(3(2) - (-1)K\right) - 2\left(2(2) - (-1)\right) + \left(2(K) - 3\right)$

$= 18 + 3K - 10 + 2K - 3$

$= 5K + 5$

Question 3 continued

b)(i) $\Pi_1 : \ r \cdot \begin{pmatrix}3\\2\\1\end{pmatrix} = 4 \qquad r = \begin{pmatrix}x\\y\\z\end{pmatrix}$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} \cdot \begin{pmatrix}3\\2\\1\end{pmatrix} = 3x + 2y + z = 4$$

(ii) $\Pi_2 : \ r = \begin{pmatrix}1\\2\\3\end{pmatrix} + \underbrace{\lambda \begin{pmatrix}1\\-1\\-1\end{pmatrix} + \mu \begin{pmatrix}1\\0\\2\end{pmatrix}}_{\text{to find normal to plane} \rightarrow d_1 \cdot d_2}$

$\boxed{\text{cartesian equation: } \begin{pmatrix}x\\y\\z\end{pmatrix} \cdot n = d}$

$$n = \left(\begin{pmatrix}1\\0\\2\end{pmatrix} \times \begin{pmatrix}1\\-1\\-1\end{pmatrix} \right) = \begin{pmatrix}2\\3\\-1\end{pmatrix}$$

$r \cdot \begin{pmatrix}2\\3\\-1\end{pmatrix} = d \qquad \leftarrow$ use point $\begin{pmatrix}1\\2\\3\end{pmatrix}$ to calculate d

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \cdot \begin{pmatrix}2\\3\\-1\end{pmatrix} = 1(2) + 2(3) + 3(-1) = 5$$

$\therefore \ 2x + 3y - z = 5$

c) If Π_1, Π_2, Π_3 form a sheaf, they will meet at a line $\therefore$ there will be infinite, consistent solutions to the simultaneous equation

$$\begin{array}{l} \Pi_1 \rightarrow \\ \Pi_2 \rightarrow \\ \Pi_3 \rightarrow \end{array} \begin{bmatrix}3 & 2 & 1\\2 & 3 & -1\\1 & k & 2\end{bmatrix} \begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}4\\5\\-1\end{bmatrix}$$

To have infinite solutions
└ the determinant $= 0$

Question 3 continued

$\det M = 5K + 5 = 0$

$K = -1$

4. A company manufacturing radios agreed a 20 year contract with a retailer to supply its radios. In the first year of the contract, 500 radios were supplied to the retailer. In each subsequent year, the number of radios supplied to the retailer was 50 more than in the previous year.

The amount received by the company for each radio during year n of the contract was £$\left(20 + \frac{n^2}{45}\right)$

The total cost of producing the radios during year n was modelled as £$(1000 + 10n^2)$

(a) Show that, according to the model, the profit made by the company in year n, £P_n, is given by

$$P_n = \frac{10}{9}(n^3 + 900n + 7200)$$

(2)

(b) Use the standard results for summations to show that the total profit made by the company in the first N years of the contract, £T_N, is given by

$$T_N = aN(N^3 + bN^2 + cN + d)$$

where a, b, c and d are constants to be found.

(5)

At the end of the 20 years, the company found that its total profit made from this contract just exceeded £500 000.

(c) Assess the model in light of this information.

(2)

4.a) In year 1, 500 radios were supplied

└ every year after year 1, 50 more are added

∴ no. of radios supplied in year n = 450 + 50n

TOTAL PROFIT = (no. radios × price for each radio) − Total cost

$= \left(20+\frac{n^2}{45}\right)(450+50n) - 1000+10n^2$

$= \left(9000+1000n+10n^2+\frac{10n^3}{9}\right) - 1000+10n^2$

$= 8000+1000n+\frac{10n^3}{9}$

$= \frac{10}{9}(n^3+900n+7200)$

Question 4 continued

b) $T_N = \sum_{n=1}^{N} P_n = \sum_{n=1}^{N}\left(\frac{10}{9}\left(n^3+900n+7200\right)\right)$

$= \frac{10}{9}\left(\sum_{n=1}^{N} n^3 + 900\sum_{n=1}^{N} n + 7200\sum_{n=1}^{N} 1\right)$

USING STANDARD SUMMATIONS

$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ $\qquad$ $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

from formulae booklet

$= \frac{10}{9}\left(\frac{N^2(N+1)^2}{4} + 900\left(\frac{N(N+1)}{2}\right) + 7200N\right)$

$= \frac{10}{9}\left(\frac{N^4+2N^3+N^2}{4} + 450\left(N^2+N\right) + 7200N\right)$

$= \frac{10}{36}\left(N^4+2N^3+N^2+1800N^2+1800N+28800N\right)$

$= \frac{10N}{36}\left(N^3+2N^2+1801N+30600\right)$

$\therefore T_N = \frac{5N}{18}\left(N^3+2N^2+1801N+30600\right)$

c) $T_{20} = \frac{5(20)}{18}\left((20)^3+2(20)^2+1801(20)+30600\right)$

$= £419\ 000 \quad (<<£500\ 000$

This predicted value is much less than the actual value $\therefore$ NOT good model

5.

$$f(z) = 8z^3 + 12z^2 + 6z + 65$$

Given that $\frac{1}{2} - i\sqrt{3}$ is a root of the equation $f(z) = 0$

(a) write down the other complex root of the equation,

(1)

(b) use algebra to solve the equation $f(z) = 0$ completely.

(3)

(c) Show the roots of $f(z)$ on a single Argand diagram.

(2)

(d) Show that the roots of $f(z)$ form the vertices of an equilateral triangle in the complex plane.

(2)

5.a) If a polynomial has real coefficients & complex solutions → they exist in complex pairs

$\therefore$ if $\frac{1}{2} - i\sqrt{3}$ is a root

then $\frac{1}{2} + i\sqrt{3}$ is also a root

b) METHOD 1 (expand known roots):

let the 3rd root be α

$$\therefore \left(z - \left(\tfrac{1}{2} - i\sqrt{3}\right)\right)\left(z - \left(\tfrac{1}{2} + i\sqrt{3}\right)\right)(z - \alpha) = f(z)$$

$$\left(z^2 - z + \tfrac{13}{4}\right)(z - \alpha) = f(z) = 8z^3 + 12z^2 + 6z + 65$$

$$z - \alpha = \left(z^2 - z + \tfrac{13}{4}\right) \overline{\big)\, 8z^3 + 12z^2 + 6z + 65}$$

$$\begin{array}{r} 8z + 20 \\ 8z^3 + 12z^2 + 6z + 65 \\ -\underline{8z^3 - 8z^2 + 26z} \\ 20z^2 - 20z + 65 \\ -\underline{20z^2 - 20z + 65} \\ 0 \end{array}$$

$$\therefore f(z) = \left(z^2 - z + \tfrac{13}{4}\right)(8z + 20)$$

Question 5 continued

$$\therefore \ 3^{rd} \text{ root} = -\frac{20}{8} = -\frac{5}{2}$$

$$\text{roots} = \frac{1}{2} - i\sqrt{3}, \ \frac{1}{2} + i\sqrt{3}, \ -\frac{5}{2}$$

METHOD 2 (using product/sum rules for polynomials):

$$ax^3 + bx^2 + cx + d = 0$$

$$\alpha + \beta + \gamma = \frac{-b}{a} \qquad \alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a} \qquad \alpha\beta\gamma = \frac{-d}{a}$$

$$f(z) = 8z^3 + 12z^2 + 6z + 65$$

$$\alpha + \beta + \gamma = -\frac{12}{8} = -\frac{3}{2} = \frac{1}{2} - \cancel{i\sqrt{3}} + \frac{1}{2} + \cancel{i\sqrt{3}} + \gamma$$

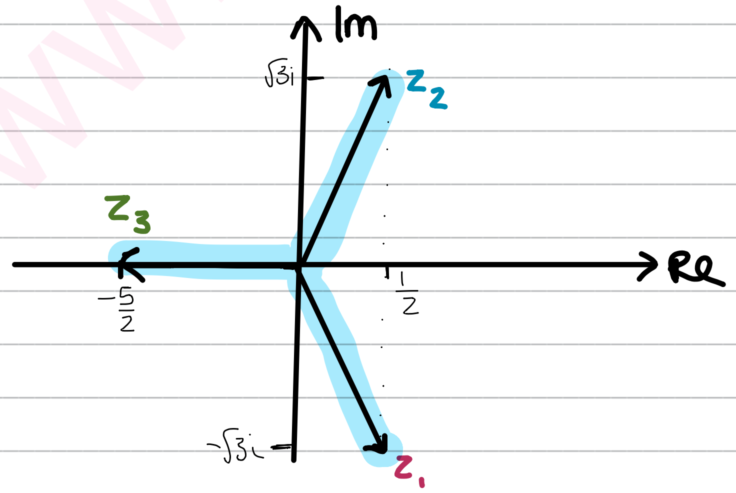
↑ 3rd root

$$-\frac{3}{2} = 1 + \gamma$$

$$\gamma = -\frac{5}{2}$$

$$\therefore \text{roots} = \frac{1}{2} - i\sqrt{3}, \ \frac{1}{2} + i\sqrt{3}, \ -\frac{5}{2}$$

c)



d) length of side z_1 to z_2 = $|z_1 - z_2|$

$= \left| \frac{1}{2} - i\sqrt{3} - \left(\frac{1}{2} + i\sqrt{3}\right) \right|$

$= |-2\sqrt{3}i| = \sqrt{(-2\sqrt{3})^2} = \sqrt{12} = 2\sqrt{3}$

$|z_1 - z_3|$

$= \left| \frac{1}{2} - i\sqrt{3} - \left(-\frac{5}{2}\right) \right|$

$= |3 - i\sqrt{3}| = \sqrt{3^2 + (\sqrt{3})^2} = \sqrt{12} = 2\sqrt{3}$

$|z_2 - z_3|$

$= \left| \frac{1}{2} + i\sqrt{3} - \left(-\frac{5}{2}\right) \right|$

$= |3 + i\sqrt{3}| = \sqrt{3^2 + (\sqrt{3})^2} = \sqrt{12} = 2\sqrt{3}$

All 3 sides of the triangle are equal in length

$\therefore$ it is equilateral

6. (a) Prove by induction that, for all $n \in \mathbb{Z}^+$

$$f(n) = n^5 + 4n$$

is divisible by 5

(6)

(b) Show that $f(-x) = -f(x)$ for all $x \in \mathbb{R}$

(1)

(c) Hence prove that $f(n)$ is divisible by 5 for all $n \in \mathbb{Z}$

(2)

INDUCTION:
- Prove true for base case
- Assume true for $n=k$
- consider $n=k+1$ & replace by assumption
- Conclusion

6.a)

Base case $n=1$:

$f(1) = 1^5 + 4(1)$

$= 1 + 4$

$= 5$ ← divisible by 5 $(5 \times 1 = 5)$

$\therefore$ statement true for $n=1$

Assume true for $n=k$

$f(k) = k^5 + 4k = 5A \qquad A \in \mathbb{Z}$ ← A is an integer

$\therefore$ assume $f(k)$ is divisible by 5

$\hookleftarrow k^5 = 5A - 4k$ (rearrange)

Consider $n=k+1$

$f(k+1) = (k+1)^5 + 4(k+1)$

$= k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 + 4k + 4$

$= \underbrace{k^5}_{\downarrow} + 5k^4 + 10k^3 + 10k^2 + 9k + 5$

$= 5A - 4k + 5k^4 + 10k^3 + 10k^2 + 9k + 5$

$= 5A + 5k^4 + 10k^3 + 10k^2 + 5k + 5$

$= 5(A + k^4 + 2k^3 + 2k^2 + k + 1)$

$= 5B \qquad B = A + k^4 + 2k^3 + 2k^2 + k + 1 \qquad \therefore B \in \mathbb{Z}$ ← integer

Question 6 continued

$\therefore f(k+1)$ is divisible by 5

hence true for $n=k+1$

Since statement is true for $n=1$, and we have proved if true for $n=k$, it is true for $n=k+1$, thus by mathematical induction, the result holds true for all positive integers

b) $f(-x)$

$= (-x)^5 + 4(-x)$

$= -x^5 - 4x$

$= -(x^5 + 4x)$

$= -f(x)$

c) - We have already proved that $f(n)$ is divisible by 5 for all positive integers $(n \in \mathbb{Z}^+)$

- We also know that $f(-n) = -f(n)$

$\therefore -f(n)$ is also divisible by 5

$\therefore$ for all positive & negative integers, n, $f(n)$ is divisible by 5

only exception to prove is $n=0$

$f(0) = 0^5 + 4(0) = 0$ $\therefore f(0)$ also divisible by 5

$0 = 5 \times 0$

(Total for Question 6 is 9 marks)

$\therefore f(n)$ divisible by 5 for all integers n

7. The population of Zebu cattle in a particular country is modelled by two sub-populations, adults and juveniles. In this model, the only factors affecting the population of the Zebu are the birth and survival rates of the population.

Data recorded in the years preceding 2018 was used to suggest the annual birth and survival rates of the population.
The results are shown in the table below, with values to 2 significant figures. It is assumed that these rates will remain the same in future years.

	Birth rate	Survival rate
Adult population	0.23	0.97
Juvenile population	0	0.87

It is also assumed that $\frac{1}{3}$ of the surviving juvenile population become adults each year.

Let A_n and J_n be the respective sub-populations, in millions, of adults and juveniles, n years after 1st January 2018. Then the adult population in year $n + 1$ satisfies the equation

$$A_{n+1} = 0.97A_n + \frac{1}{3}(0.87)J_n = 0.97A_n + 0.29J_n$$

(a) Form the corresponding equation for the juvenile population in year $n + 1$ under this model, justifying your values.

(2)

The total population on 1st January 2018 was estimated, to 2 significant figures, as 1.5 million Zebu, with 1.2 million of these being adults.

(b) Find the value of p and the matrix $\mathbf{M}$ such that the population of Zebu can be modelled by the system

$$\begin{pmatrix} A_0 \\ J_0 \end{pmatrix} = \begin{pmatrix} 1.2 \\ p \end{pmatrix} \qquad \begin{pmatrix} A_{n+1} \\ J_{n+1} \end{pmatrix} = \mathbf{M}\begin{pmatrix} A_n \\ J_n \end{pmatrix}$$

Give p to 2 significant figures and each entry of $\mathbf{M}$ to 2 decimal places.

(3)

Using the model formed in (b), find, to 3 significant figures,

(c) (i) the **total** Zebu population that was present on 1st January 2017

(ii) the predicted **juvenile** Zebu population on 1st January 2025

(5)

As a result of the predictions of this model, it is decided that the country will export 15 000 juveniles to a neighbouring country at the end of each year.

(d) Adapt the model from 2018 onwards to include this export.

(2)

(e) State one limitation of this model.

(1)

7.a) $J_{n+1} = 0.23A_n + \frac{2}{3}(0.87)J_n$

0.23 is the birth rate ∴ 0.23 × adult population = new juvenile population

If $\frac{1}{3}$ of the surviving juvenile population become adults ∴ $(1-\frac{1}{3})$ remain juvenile the next year

$$J_{n+1} = 0.23A_n + 0.58J_n$$

b) $\begin{pmatrix}A_0\\J_0\end{pmatrix} = \begin{pmatrix}1.2\\1.5-1.2\end{pmatrix} = \begin{pmatrix}1.2\\0.3\end{pmatrix}$

$p = 0.30$

$$\begin{pmatrix}A_{n+1}\\J_{n+1}\end{pmatrix} = M\begin{pmatrix}A_n\\J_n\end{pmatrix}$$

$A_{n+1} = 0.97A_n + 0.29J_n$

$J_{n+1} = 0.23A_n + 0.58J_n$

$$\therefore M = \begin{bmatrix}0.97 & 0.29\\0.23 & 0.58\end{bmatrix}$$

c) (i) $\begin{pmatrix}A_{2018}\\J_{2018}\end{pmatrix} = M\begin{pmatrix}A_{2017}\\J_{2017}\end{pmatrix}$

$$M^{-1}\begin{pmatrix}A_{2018}\\J_{2018}\end{pmatrix} = \cancel{M^{-1}}\cancel{M}\begin{pmatrix}A_{2017}\\J_{2017}\end{pmatrix}$$

$$M^{-1}\begin{pmatrix}A_{2018}\\J_{2018}\end{pmatrix} = \begin{bmatrix}0.97 & 0.29\\0.23 & 0.58\end{bmatrix}^{-1}\begin{bmatrix}1.2\\0.3\end{bmatrix}$$

$$= \begin{bmatrix}1.228\\0.0302\end{bmatrix} = \begin{bmatrix}A_{2017}\\J_{2017}\end{bmatrix}$$

Question 7 continued

Total pop $= (1.228 + 0.0302)$ million

$= 1.26$ million

(ii) $\begin{pmatrix} A_{2025} \\ J_{2025} \end{pmatrix} = M^7 \begin{pmatrix} A_{2018} \\ J_{2018} \end{pmatrix}$

$= \begin{bmatrix} 0.97 & 0.29 \\ 0.23 & 0.58 \end{bmatrix}^7 \begin{bmatrix} 1.2 \\ 0.3 \end{bmatrix}$

$= \begin{bmatrix} 2.117 \\ 0.938 \end{bmatrix}$

$\therefore J_{2025} = 0.938$ million

938 000

d) $\begin{bmatrix} A_{n+1} \\ J_{n+1} \end{bmatrix} = M \begin{bmatrix} A_n \\ J_n \end{bmatrix} - \begin{bmatrix} 0 \\ 0.015 \end{bmatrix}$

e) The exportation might have an effect on the proportion of juveniles becoming adults each year

8.

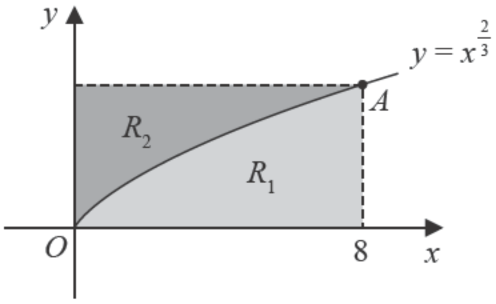


Figure 1

Figure 1 shows a sketch of the curve with equation $y = x^{\frac{2}{3}}, x \geqslant 0$

The curve passes through the point A with x coordinate 8
The region R_1 is bounded by the curve, the vertical line passing through A and the x-axis.
The region R_2 is bounded by the curve, the horizontal line passing through A and the y-axis.

The solid V_1 is formed by rotating the region R_1 through 360° about the x-axis.
The solid V_2 is formed by rotating the region R_2 through 360° about the y-axis.

(a) Show that the exact volume of the solid V_1 is $\frac{384\pi}{7}$

(4)

The solids V_1 and V_2 are placed in an empty container. A solid is selected at random and then replaced in the container. This is carried out 10 times.

Given that the probability of selecting each type of solid is proportional to its volume,

(b) find, to 4 decimal places, the probability that the solid V_2 is selected exactly 8 times.

(7)

8.a) $V_{\substack{\text{about}\\ x\text{-axis}}} = \pi\int_a^b y^2 \, dx$

$y = x^{\frac{2}{3}}$

$V_1 = \pi\int_0^8 (x^{2/3})^2 \, dx$

$V_1 = \pi\int_0^8 x^{4/3} \, dx$

$V_1 = \pi\left[\frac{3}{7}x^{7/3}\right]_0^8 = \pi\left(\frac{3}{7}\times(8)^{\frac{7}{3}}\right) = \frac{384\pi}{7}$

Question 8 continued

b) $V_{\text{about } y \text{ axis}} = \pi \int_a^b x^2 \, dy$

$y = x^{\frac{2}{3}}$

$x = y^{\frac{3}{2}}$

limits → when $x = 8$

$y = (8)^{2/3} = 4$

→ when $x = 0$

$y = 0$

$$V_2 = \pi \int_0^4 (y^{3/2})^2 \, dy$$

$$V_2 = \pi \int_0^4 y^3 \, dy$$

$$V_2 = \pi \left[\frac{y^4}{4} \right]_0^4 = \pi \left(\frac{4^4}{4} - \frac{0^4}{4} \right) = 64\pi$$

Given that the probability of selecting the solid is proportional to its volume

$$\text{prob of selecting } V_2 = \frac{64\cancel{\pi}}{64\cancel{\pi} + \frac{384\cancel{\pi}}{7}} = \frac{7}{13}$$

∴ prob of selecting V_2 8 times out of 10 times

$X \sim B(10, \frac{7}{13})$

total no. of trials ↑

← prob of picking V_2

$$P(X = 8) = {}^{10}C_8 \left(\frac{7}{13}\right)^8 \left(\frac{6}{13}\right)^2 = 0.0677$$

9. A small comet C is passing near to a planet. The planet can be modelled as a sphere with centre O taken as a fixed point in space, so that the motion of the comet is relative to the origin O.

The diameter of the planet is 13 000 km.

The comet is monitored by satellites orbiting the planet.

When the monitoring begins the comet is at position $146\mathbf{i} + 234\mathbf{j} - 85\mathbf{k}$ and is moving with vector $-21\mathbf{i} - 33\mathbf{j} + 13\mathbf{k}$ every hour, where the units are in thousands of kilometres.

Assuming the comet maintains a straight line course throughout its motion,

(a) determine whether or not the comet will collide with the planet.

(6)

Two of the satellites, A and B, have position vectors $\overrightarrow{OA} = 5\mathbf{i} + 12\mathbf{k}$ and $\overrightarrow{OB} = 4\mathbf{i} + 12\mathbf{j} - 3\mathbf{k}$ at the beginning of monitoring. They return to these positions every 4 hours.

(b) Find the expected angle ACB between the comet and the satellites A and B when they first return to their initial positions. Give your answer to the nearest 0.1°

(4)

(c) Give a reason why the answer to (b) may differ from the true value.

(1)

9.a) Path of comet : $r = \begin{pmatrix}146\\234\\-85\end{pmatrix} + \lambda\begin{pmatrix}-21\\-33\\13\end{pmatrix}$

Distance $\overrightarrow{OC}$

$= \begin{pmatrix}146\\234\\-85\end{pmatrix} + \lambda\begin{pmatrix}-21\\-33\\13\end{pmatrix} - \begin{pmatrix}0\\0\\0\end{pmatrix}$

Shortest distance is when $\overrightarrow{OC}$ is perpendicular to the direction vector of the comet

if $a \cdot b = 0$ a,b are perpendicular

$$\underset{\overrightarrow{OC}}{\begin{pmatrix}146-21\lambda\\234-33\lambda\\-85+13\lambda\end{pmatrix}} \cdot \underset{\text{direction vector}}{\begin{pmatrix}-21\\-33\\13\end{pmatrix}} = 0$$

$-21(146-21\lambda)-33(234-33\lambda)+13(-85+13\lambda) = 0$

$-11893 + 1699\lambda = 0$ ∴ shortest distance occurs when $\lambda = 7$

Question 9 continued

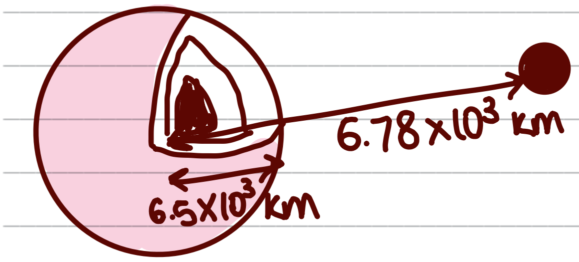
$$\therefore OC_{min} = \begin{pmatrix}146-21\lambda\\234-33\lambda\\-85+13\lambda\end{pmatrix} = \begin{pmatrix}146-21(7)\\234-33(7)\\-85+13(7)\end{pmatrix} = \begin{pmatrix}-1\\3\\6\end{pmatrix}$$

$$\therefore |OC_{min}| = \sqrt{(-1)^2+(3)^2+(6)^2}$$

$$= \sqrt{46}$$

∴ Minimum distance of comet from centre of planet

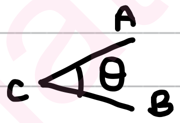
$= 6.78 \times 10^3$ km



$\therefore (6.78\times10^3) > (6.5\times10^3)$

so comet misses the planet

b) $\boxed{\cos\theta = \frac{a\cdot b}{|a||b|}}$



The satellites first return to their initial positions 4 hours later

∴ Position of comet when $\lambda = 4$

$$\text{Path of comet}: r = \begin{pmatrix}146\\234\\-85\end{pmatrix} + 4\begin{pmatrix}-21\\-33\\13\end{pmatrix}$$

$$\vec{OC} = \begin{pmatrix}62\\102\\-33\end{pmatrix}$$

$$\vec{AC} = \vec{OC} - \vec{OA} = \begin{pmatrix}62\\102\\-33\end{pmatrix} - \begin{pmatrix}5\\0\\12\end{pmatrix} = \begin{pmatrix}57\\102\\-45\end{pmatrix}$$

Question 9 continued

$$\vec{BC} = \vec{OC} - \vec{OB} = \begin{pmatrix}62\\102\\-33\end{pmatrix} - \begin{pmatrix}4\\12\\-3\end{pmatrix} = \begin{pmatrix}58\\90\\-30\end{pmatrix}$$

$$\cos\theta = \frac{\begin{pmatrix}57\\102\\-45\end{pmatrix}\cdot\begin{pmatrix}58\\90\\-30\end{pmatrix}}{\left|\begin{pmatrix}57\\102\\-45\end{pmatrix}\right|\left|\begin{pmatrix}58\\90\\-30\end{pmatrix}\right|} = \frac{13836}{\sqrt{15678}\sqrt{12364}} = 0.9937$$

$$\theta = 6.399..^\circ = 6.4^\circ$$

c) The comet may not follow a straight line exactly